EL ROTACIONAL DE UN CAMPO VECTORIAL EN COORDENADAS RECTÁNGULARES, ESFÉRICAS Y CILÍNDRICAS:

Del teorema del rotor de Stokes, se tiene:

$$\int_S rot(\vec{f}).d\vec{S} = \oint_L \vec{f}.d\vec{l}$$

o bien:

$$\int_s \left\{ \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right).d\vec{y} \wedge d\vec{z} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right).d\vec{z} \wedge d\vec{x} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right).d\vec{x} \wedge d\vec{y} \right\} =$$

$$= \oint_L f_1.dx + f_2dy + f_3dz$$

que, para coordenadas generales podemos escribir la forma diferencial como:

$$\boxed{\begin{aligned} rot\vec{t}(\vec{f}) = (&\frac{1}{ds_{q_1}}\left[\frac{\partial(f_3.dr_3)}{\partial q_2}.dq_2 - \frac{\partial(f_2.dr_2)}{\partial q_3}dq_3 \right], \\ &\frac{1}{ds_{q_2}}\left[\frac{\partial(f_1.dr_1)}{\partial q_3}.dq_3 - \frac{\partial(f_3.dr_3)}{\partial q_1}dq_1 \right], \\ &\frac{1}{ds_{q_3}}\left[\frac{\partial(f_2.dr_2)}{\partial q_1}.dq_1 - \frac{\partial(f_1.dr_1)}{\partial q_2}dq_2 \right]) \end{aligned}}$$

Expresión en coordenadas rectangulares o cartesianas:

En estas coordenadas es: $dS_{q_1} = dy.dz,\ dS_{q_2} = dx.dz,\ dS_{q_3} = dxdy$

Y también es $dr_1 = dx,\ dr_2 = dy,\ dr_3 = dz$

Por tanto:

$$\begin{aligned} rot\vec{t}(\vec{f}) = (&\frac{1}{dy.dz}\left[\frac{\partial(f_3.dz)}{\partial y}.dy - \frac{\partial(f_2.dy)}{\partial z}dz \right], \\ &\frac{1}{dx.dz}\left[\frac{\partial(f_1.dx)}{\partial z}.dz - \frac{\partial(f_3.dz)}{\partial x}dx \right], \\ &\frac{1}{dx.dy}\left[\frac{\partial(f_2.dy)}{\partial x}.dx - \frac{\partial(f_1.dx)}{\partial y}dy \right]) \end{aligned}$$

O sea:

$$\boxed{rot\vec{t}(\vec{f}) = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right)\vec{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right)\vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)\vec{k}}$$

Expresión en coordenadas esféricas:

En estas coordenadas es:

$$dS_{q_1} = \rho^2.sen\theta.d\theta.d\phi,\quad dS_{q_2} = \rho.sen\theta.d\rho.d\phi,\quad dS_{q_3} = \rho.d\rho.d\theta$$

Y también es $dr_1 = d\rho$, $dr_2 = \rho.d\theta$, $dr_3 = \rho.sen\theta.d\phi$

Por tanto:

$$rot\vec{t}(\vec{f}) = \left(\frac{1}{\rho^2.sen\theta.d\theta.d\phi}\left[\frac{\partial(f_3.\rho.sen\theta.d\phi)}{\partial\theta}.d\theta - \frac{\partial(f_2.\rho.d\theta)}{\partial\phi}d\phi\right],\right.$$
$$\frac{1}{\rho.sen\theta.d\rho.d\phi}\left[\frac{\partial(f_1.d\rho)}{\partial\phi}.d\rho - \frac{\partial(f_3.\rho.sen\theta.d\phi)}{\partial\rho}d\rho\right],$$
$$\left.\frac{1}{\rho.d\rho.d\theta}\left[\frac{\partial(f_2.\rho.d\theta)}{\partial\rho}.d\rho - \frac{\partial(f_1.d\rho)}{\partial\theta}d\theta\right]\right)$$

O sea:

$$\boxed{rot\vec{t}(\vec{f}) = \frac{1}{\rho.sen\theta}\left[\frac{\partial(f_3.sen\theta)}{\partial\theta} - \frac{\partial(f_2)}{\partial\phi}\right].\vec{\rho}^{\,o} + \frac{1}{\rho}\left[\frac{1}{sen\theta}\frac{\partial(f_1.d\rho)}{\partial\phi} - \frac{\partial(f_3.\rho.)}{\partial\rho}\right].\vec{\theta}^{\,o} + \frac{1}{\rho}\left[\frac{\partial(f_2.\rho.)}{\partial\rho} - \frac{\partial(f_1)}{\partial\theta}\right].\vec{\phi}^{\,o}}$$

Expresión en coordenadas cilíndricas:

En estas coordenadas es:

$$dS_{q_1} = \rho.d\phi.dh \quad dS_{q_2} = d\rho.dh \quad dS_{q_3} = \rho.d\rho.d\phi$$

Y también es $dr_1 = d\rho$, $dr_2 = \rho.d\phi$, $dr_3 = dh$

Por tanto:

$$rot\vec{t}(\vec{f}) = \left(\frac{1}{\rho.d\phi.dh}\left[\frac{\partial(f_3.dh)}{\partial\phi}d\phi - \frac{\partial(f_2.\rho.d\phi)}{\partial h}dh\right],\right.$$
$$\frac{1}{d\rho.dh}\left[\frac{\partial(f_1.d\rho)}{\partial h}.dh - \frac{\partial(f_3.dh)}{\partial\rho}d\rho\right],$$
$$\left.\frac{1}{\rho.d\rho.d\phi}\left[\frac{\partial(f_2.\rho.d\phi)}{\partial\rho}.d\rho - \frac{\partial(f_1.d\rho)}{\partial\phi}d\phi\right]\right)$$

Y queda finalmente:

$$rot(\vec{f}) = \left[\frac{1}{\rho}\frac{\partial(f_3)}{\partial\phi} - \frac{\partial(f_2)}{\partial h}\right].\vec{\rho}^{\,o} + \left[\frac{\partial(f_1)}{\partial h} - \frac{\partial(f_3)}{\partial\rho}\right].\vec{\phi}^{\,o} + \frac{1}{\rho}\left[\frac{\partial(f_2.\rho)}{\partial\rho} - \frac{\partial(f_1)}{\partial\phi}\right].\vec{h}^{\,o}$$

En resumen:

En cartesianas:

$$rot(\vec{f}) = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\vec{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\vec{k}$$

En esféricas:

$$rot(\vec{f}) = \frac{1}{\rho.sen\theta}\left[\frac{\partial(f_3.sen\theta)}{\partial\theta} - \frac{\partial(f_2)}{\partial\phi}\right].\vec{\rho}^{\,o} + \frac{1}{\rho}\left[\frac{1}{sen\theta}\frac{\partial(f_1.d\rho)}{\partial\phi} - \frac{\partial(f_3.\rho.)}{\partial\rho}\right].\vec{\theta}^{\,o} +$$
$$+ \frac{1}{\rho}\left[\frac{\partial(f_2.\rho.)}{\partial\rho} - \frac{\partial(f_1)}{\partial\theta}\right].\vec{\phi}^{\,o}$$

En cilíndricas:

$$rot(\vec{f}) = \left[\frac{1}{\rho}\frac{\partial(f_3)}{\partial\phi} - \frac{\partial(f_2)}{\partial h}\right].\vec{\rho}^{\,o} + \left[\frac{\partial(f_1)}{\partial h} - \frac{\partial(f_3)}{\partial\rho}\right].\vec{\phi}^{\,o} + \frac{1}{\rho}\left[\frac{\partial(f_2.\rho)}{\partial\rho} - \frac{\partial(f_1)}{\partial\phi}\right].\vec{h}^{\,o}$$