Título:

EL PRODUCTO DE EULER. DEMOSTRACIÓN

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INTRODUCCIÓN.

La demostración de la veracidad del producto de Euler que muestro a continuación está basada exclusivamente en una relación que obtuve entre la serie $\sum_{n=0}^{n=\infty}\left(\frac{1}{(2n+1)^Z}\right)$ de la suma de los inversos de los números impares elevados a un número complejo con parte real mayor que cero, y la función que llamo $\beta(z)$ que defino en el trabajo.
No puedo hacer referencia a propiedades o conocimientos establecidos, porque la propiedad citada anteriormente no se basa en nada anterior.

Sin más preámbulo comienzo.

I. **EXPRESIONES GENERALES**.

En todas las expresiones es, salvo mención explícita:

1) $z = x + i\,y \;,\; x,y \;\in IR \;.\; x \rangle 0 \;,\; i = \sqrt{(-1)}$

2) $IP =$ conjunto de los números primos.

3) $P_1, P_2, \ldots\ldots\ldots\ldots, P_m \quad \in IP - \{2\}$
$P_1 \neq P_2 \neq .. \neq P_m$

$$\Sigma_{P_1} = \sum_{p=3}^{p=+\infty} \frac{1}{p^Z} = \frac{1}{3^Z} + \frac{1}{5^Z} + \frac{1}{7^Z} + \frac{1}{11^Z} + \frac{1}{13^Z} + \frac{1}{17^Z} + \ldots\ldots\ldots\ldots \infty$$

$$\Sigma_{P_2} = \sum_{p=3}^{p=+\infty} \frac{1}{p^{2Z}} = \frac{1}{3^{2Z}} + \frac{1}{5^{2Z}} + \frac{1}{7^{2Z}} + \frac{1}{11^{2Z}} + \frac{1}{13^{2Z}} + \frac{1}{17^{2Z}} + \ldots\ldots\ldots\ldots \infty$$

$$\Sigma_{P_m} = \sum_{p=3}^{p=+\infty} \frac{1}{p^{mZ}} = \frac{1}{3^{mZ}} + \frac{1}{5^{mZ}} + \frac{1}{7^{mZ}} + \frac{1}{11^{mZ}} + \frac{1}{13^{mZ}} + \frac{1}{17^{mZ}} + \ldots\ldots\ldots\ldots \infty$$

$$\Sigma_2 = \frac{1}{2!} \sum_{P_1,P_2=3}^{P_1,P_2=+\infty} \frac{1}{P_1^Z\, P_2^Z} = \frac{1}{3^Z\,5^Z} + \frac{1}{3^Z\,7^Z} + \frac{1}{3^Z\,11^Z} + \ldots\ldots\ldots\ldots + \frac{1}{5^Z\,7^Z} + \frac{1}{5^Z\,11^Z} + \frac{1}{5^Z\,13^Z} + \ldots\ldots\ldots$$

$$+ \frac{1}{7^Z\,11^Z} + \frac{1}{7^Z\,13^Z} + \frac{1}{7^Z\,17^Z} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{11^Z\,13^Z} + \frac{1}{11^Z\,17^Z} + \frac{1}{11^Z\,19^Z} +$$

$$\Sigma_3 = \frac{1}{3!} \sum_{P_1,P_2,P_3=3}^{P_1,P_2,P_3=+\infty} \frac{1}{P_1^Z P_2^Z P_3^Z} = \frac{1}{3^Z 5^Z 7^Z} + \frac{1}{3^Z 5^Z 11^Z} + \frac{1}{3^Z 5^Z 13^Z} +..+ \frac{1}{3^Z 7^Z 11^Z} + \frac{1}{3^Z 7^Z 13^Z} + \frac{1}{3^Z 7^Z 17^Z} +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{5^Z 7^Z 11^Z} + \frac{1}{5^Z 7^Z 17^Z} + \frac{1}{5^Z 7^Z 19^Z} + \ldots\ldots\ldots\infty$$

$$\Sigma_m = \frac{1}{m!} \sum_{P_1,P_2,\ldots,P_m=3}^{P_1,P_2,\ldots,P_m=+\infty} \frac{1}{P_1^Z P_2^Z \ldots\ldots P_m^Z}$$

II. RELACIONES ENTRE LAS EXPRESIONES GENERALES.

$$\Sigma_2 = \frac{1}{2!} \sum_{P_1,P_2=3}^{P_1,P_2=+\infty} \frac{1}{P_1^Z P_2^Z} = \frac{1}{2!} \sum_{P_1=3}^{P_1=+\infty} \left(\frac{1}{P_1^Z} \sum_{P_2=3;P_2\neq P_1}^{P_2=+\infty} \frac{1}{P_2^Z} \right) = \frac{1}{2!} \sum_{P_1=3}^{P_1=+\infty} \left(\frac{1}{P_1^Z} \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} - \frac{1}{P_1^Z} \right) \right) =$$

$$= \frac{1}{2!} \left(\left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \right) \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \right) - \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{2Z}} \right) = \frac{1}{2!} \left(\left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \right)^2 - \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{2Z}} \right) =$$

$$= \frac{1}{2!} \left((\Sigma_{P_1})^2 - \Sigma_{P_2} \right). = \frac{(\Sigma_{P_1})^2}{2!} - \frac{\frac{\Sigma_{P_2}}{2}}{1!}$$

$$\Sigma_3 = \frac{1}{3!} \sum_{P_1,P_2,P_3=3}^{P_1,P_2,P_3=+\infty} \frac{1}{P_1^Z P_2^Z P_3^Z} = \frac{1}{3!} \sum_{P_1=3}^{P_1=+\infty} \left(\frac{1}{P_1^Z} \sum_{P_2,P_3=3;P_2,P_3\neq P_1}^{P_2,P_3=+\infty} \frac{1}{P_2^Z P_3^Z} \right) =$$

$$= \frac{1}{3!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left(\left(\sum_{P_2=3,P_2\neq P_1}^{P_2=+\infty} \frac{1}{P_2^Z} \right)^2 - \sum_{P_2=3,P_2\neq P_1}^{P_2=+\infty} \frac{1}{P_2^{2Z}} \right) =$$

$$= \frac{1}{3!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left(\left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} - \frac{1}{P_1^Z} \right)^2 - \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{2Z}} - \frac{1}{P_1^{2Z}} \right) \right) =$$

$$= \frac{1}{3!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left((\Sigma_{P_1})^2 - \frac{2}{P_1^Z} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} + \frac{2}{P_1^{2Z}} - \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{2Z}} \right) =$$

$$= \frac{1}{3!}\left((\Sigma_{P_1})^2 \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \right) - 2\,\Sigma_{P_1} \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{2Z}} \right) + 2 \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{3Z}} - \Sigma_{P_2} \left(\sum_{P_1=3_2}^{P_1=+\infty} \frac{1}{P_1^Z} \right) \right) =$$

$$= \frac{1}{3!}\left((\Sigma_{P_1})^3 - 3\,\Sigma_{P_1}\Sigma_{P_2} + 2\,\Sigma_{P_3} \right). = \frac{\left(\Sigma_{P_1}\right)^3}{3!} - \frac{\Sigma_{P_1}}{1!} \quad \frac{\frac{\Sigma_{P_2}}{2}}{1!} + \frac{\frac{\Sigma_{P_3}}{3}}{1!}$$

$$\Sigma_4 = \frac{1}{4!} \sum_{P_1,P_2,P_3,P_4=3}^{P_1,P_2,P_3,P_4=+\infty} \frac{1}{P_1^Z\, P_2^Z\, P_3^Z\, P_4^Z} = \frac{1}{4!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left(\sum_{P_2,P_3,P_4=3;P_2,P_3,P_4\neq P_1}^{P_2,P_3,P_4=+\infty} \frac{1}{P_2^Z\, P_3^Z\, P_4^Z} \right) =$$

$$= \frac{1}{4!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left(\left(\sum_{P_2=3P_2\neq P_1}^{P_2=+\infty} \frac{1}{P_2^Z} \right)^3 - 3 \left(\sum_{P_2=3P_2\neq P_1}^{P_2=+\infty} \frac{1}{P_2^Z} \right) \left(\sum_{P_2=3P_2\neq P_1}^{P_2=+\infty} \frac{1}{P_2^{2Z}} \right) + 2 \sum_{P_2=3P_2\neq P_1}^{P_2=+\infty} \frac{1}{P_2^{3Z}} \right) =$$

$$\frac{1}{4!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left(\left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} - \frac{1}{P_1^Z} \right)^3 - 3 \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} - \frac{1}{P_1^Z} \right) \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{2Z}} - \frac{1}{P_1^{2Z}} \right) + 2 \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{3Z}} - \frac{1}{P_1^{3Z}} \right)$$

$$= \frac{1}{4!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left((\Sigma_{P_1})^3 - \frac{3}{P_1^Z} (\Sigma_{P_1})^2 + \frac{6}{P_1^{2Z}} \Sigma_{P_1} - \frac{6}{P_1^{3Z}} + \frac{3}{P_1^Z} \Sigma_{P_2} - 3\,\Sigma_{P_1}\Sigma_{P_2} + 2\,\Sigma_{P_3} \right) =$$

$$= \frac{1}{4!}\left((\Sigma_{P_1})^4 - 3(\Sigma_{P_1})^2\,\Sigma_{P_2} + 6\,\Sigma_{P_1}\Sigma_{P_3} - 6\,\Sigma_{P_4} + 3(\Sigma_{P_2})^2 - 3(\Sigma_{P_1})^2\,\Sigma_{P_2} + 2\,\Sigma_{P_1}\Sigma_{P_3} \right) =$$

$$= \frac{1}{4!}\left((\Sigma_{P_1})^4 - 6(\Sigma_{P_1})^2\,\Sigma_{P_2} + 8\,\Sigma_{P_1}\Sigma_{P_3} - 6\,\Sigma_{P_4} + 3(\Sigma_{P_2})^2 \right) =$$

$$= \frac{\left(\Sigma_{P_1}\right)^4}{4!} - \frac{\left(\Sigma_{P_1}\right)^2}{2!} \quad \frac{\frac{\Sigma_{P_2}}{2}}{1!} + \frac{\Sigma_{P_1}}{1!} \quad \frac{\frac{\Sigma_{P_3}}{3}}{1!} - \frac{\frac{\Sigma_{P_4}}{4}}{1!} + \frac{\left(\frac{\Sigma_{P_2}}{2}\right)^2}{2!}$$

$$\Sigma_5 = \frac{1}{5!} \sum_{P_1,P_2,P_3,P_4,P_5=3}^{P_1,P_2,P_3,P_4,P_5=+\infty} \left(\frac{1}{P_1^Z\, P_2^Z\, P_3^Z\, P_4^Z\, P_5^Z} \right) = \frac{1}{5!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left(\sum_{P_2,P_3,P_4,P_5=3;P_2,P_3,P_4\neq P_1}^{P_2,P_3,P_4,P_5=+\infty} \frac{1}{P_2^Z\, P_3^Z\, P_4^Z\, P_5^Z} \right) =$$

$$= \frac{1}{5!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left(\begin{array}{c} \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} - \frac{1}{P_1^Z} \right)^4 - 6\left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} - \frac{1}{P_1^Z} \right)^2 \left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{2Z}} - \frac{1}{P_1^{2Z}} \right) + \\ +8\left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} - \frac{1}{P_1^Z} \right)\left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{3Z}} - \frac{1}{P_1^{3Z}} \right) - 6\left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{4Z}} - \frac{1}{P_1^{4Z}} \right) + \\ +3\left(\sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^{2Z}} - \frac{1}{P_1^{2Z}} \right)^2 \end{array} \right) =$$

$$= \frac{1}{5!} \sum_{P_1=3}^{P_1=+\infty} \frac{1}{P_1^Z} \left(\begin{array}{c} (\Sigma_{P_1})^4 - \frac{4}{P_1^Z}(\Sigma_{P_1})^3 + \frac{12}{P_1^{2Z}}(\Sigma_{P_1})^2 - \frac{24}{P_1^{3Z}}\Sigma_{P_1} + \frac{24}{P_1^{4Z}} - 6(\Sigma_{P_1})^2\, \Sigma_{P_2} + \\ +\frac{12}{P_1^Z}\Sigma_{P_1}\Sigma_{P_2} - \frac{12}{P_1^{2Z}}\Sigma_{P_2} + 8\,\Sigma_{P_1}\Sigma_{P_3} - \frac{8}{P_1^Z}\Sigma_{P_3} - 6\,\Sigma_{P_4} + 3(\Sigma_{P_2})^2 \end{array} \right) =$$

$$= \frac{1}{5!}\left((\Sigma_{P_1})^5 - 10(\Sigma_{P_1})^3\Sigma_{P_2} + 20(\Sigma_{P_1})^2\Sigma_{P_3} - 30\,\Sigma_{P_1}\Sigma_{P_4} + 24\,\Sigma_{P_5} - 20\,\Sigma_{P_2}\Sigma_{P_3} + 15\,\Sigma_{P_1}(\Sigma_{P_2}) \right)$$

$$= \frac{(\Sigma_{P_1})^5}{5!} - \frac{(\Sigma_{P_1})^3}{3!}\ \frac{\frac{\Sigma_{P_2}}{2}}{1!} + \frac{(\Sigma_{P_1})^2}{2!}\ \frac{\frac{\Sigma_{P_3}}{3}}{1!} - \frac{\Sigma_{P_1}}{1!}\ \frac{\frac{\Sigma_{P_4}}{4}}{1!} + \frac{\frac{\Sigma_{P_5}}{5}}{1!} - \frac{\frac{\Sigma_{P_2}}{2}}{1!}\ \frac{\frac{\Sigma_{P_3}}{3}}{1!} + \frac{\Sigma_{P_1}}{1!}\ \frac{\left(\frac{\Sigma_{P_2}}{2}\right)^2}{2!}$$

Del mismo modo se obtienen los siguientes: $\Sigma_6, \Sigma_7, \ldots\ldots\ldots\ldots,$ si bien no es difícil intuir,tras los resultados anteriores,que los términos pueden obtenerse de forma sistemática a partir de los elementos adecuados $\Sigma_{p_1}, \Sigma_{p_2}, \Sigma_{p_3}, \ldots$ y las potencias adecuadas de éstos.

Los términos de los sumandos de Σ_6, sus signos y sus coeficientes se obtinen así:

$(\Sigma_{P_1})^6$..*signo* : $(-1)^{6}$ ....*coeficiente* : $\frac{(\Sigma_{P_1})^6}{6!}$

$(\Sigma_{P_1})^4\Sigma_{P_2}$.................................*signo* : $(-1)^{4+1}$..........*coeficiente* : $\frac{(\Sigma_{P_1})^4}{4!}\frac{\frac{\Sigma_{P_2}}{2}}{1!}$

$(\Sigma_{P_1})^3\Sigma_{P_3}$.................................*signo* : $(-1)^{3+1}$...........*coeficiente* : $\frac{(\Sigma_{P_1})^3}{3!}\frac{\frac{\Sigma_{P_3}}{3}}{1!}$

$(\Sigma_{P_1})^2(\Sigma_{P_2})^2$..........................*signo* : $(-1)^{2+2}$............*coeficiente* : $\frac{(\Sigma_{P_1})^2}{2!}\frac{\left(\frac{\Sigma_{P_2}}{2}\right)^2}{2!}$

$(\Sigma_{P_1})^2\Sigma_{P_4}$................................*signo* : $(-1)^{2+1}$...........*coeficiente* : $\frac{(\Sigma_{P_1})^2}{2!}\frac{\frac{\Sigma_{P_4}}{4}}{1!}$

$\Sigma_{P_1}\Sigma_{P_2}\Sigma_{P_3}$..............................*signo* : $(-1)^{1+1+1}$........*coeficiente* : $\frac{\Sigma_{P_1}}{1!}\frac{\frac{\Sigma_{P_2}}{2}}{1!}\frac{\frac{\Sigma_{P_3}}{3}}{1!}$

$\Sigma_{P_1}\Sigma_{P_5}$......................................*signo* : $(-1)^{1+1}$...........*coeficiente* : $\frac{\Sigma_{P_1}}{1!}\frac{\frac{\Sigma_{P_5}}{5}}{1!}$...

$\Sigma_{P_2}\Sigma_{P_4}$.......................................*signo* : $(-1)^{1+1}$..........*coeficiente* : $\frac{\frac{\Sigma_{P_2}}{2}}{1!}\frac{\frac{\Sigma_{P_4}}{4}}{1!}$

$(\Sigma_{P_2})^3$..*signo* : $(-1)^3$............*coeficiente* : $\frac{\left(\frac{\Sigma_{P_2}}{2}\right)^3}{3!}$

$(\Sigma_{P_3})^2$..*signo* : $(-1)^2$............*coeficiente* : $\frac{\left(\frac{\Sigma_{P_3}}{3}\right)^2}{2!}$

Σ_{P_6}..*signo* : $(-1)^1$..........*coeficiente* : $\frac{\frac{\Sigma_{P_6}}{6}}{1!}$

Para los términos impares, $\Sigma_7, \Sigma_9, \ldots.$ los coeficientes se obtienen igual, y el signo

de cada coeficiente se obtiene igual, pero sumando 1 al exponente.

Los términos de los sumandos de Σ_7, sus signos y sus coeficientes se obtinen así:

$(\Sigma_{P_1})^7$..............................*signo* : $(-1)^{7+1}$................*coeficiente* : $\frac{(\Sigma_{P_1})^7}{7!}$

$(\Sigma_{P_1})^5\,\Sigma_{P_2}$......................*signo* : $(-1)^{5+1+1}$................*coeficiente* : $\frac{(\Sigma_{P_1})^5}{5!}\,\frac{\frac{\Sigma_{P_2}}{2}}{1!}$

$(\Sigma_{P_1})^4\,\Sigma_{P_3}$......................*signo* : $(-1)^{4+1+1}$................*coeficiente* : $\frac{(\Sigma_{P_1})^4}{4!}\,\frac{\frac{\Sigma_{P_3}}{3}}{1!}$

$(\Sigma_{P_1})^3(\Sigma_{P_2})^2$..................*signo* : $(-1)^{3+2+1}$................*coeficiente* : $\frac{(\Sigma_{P_1})^3}{3!}\,\frac{\left(\frac{\Sigma_{P_2}}{2}\right)^2}{2!}$

$(\Sigma_{P_1})^3\,\Sigma_{P_4}$......................*signo* : $(-1)^{3+1+1}$.................*coeficiente* : $\frac{(\Sigma_{P_1})^3}{3!}\,\frac{\frac{\Sigma_{P_4}}{4}}{1!}$

$(\Sigma_{P_1})^2\,\Sigma_{P_2}\Sigma_{P_3}$................*signo* : $(-1)^{2+1+1+1}$..............*coeficiente* : $\frac{(\Sigma_{P_1})^2}{2!}\,\frac{\frac{\Sigma_{P_2}}{2}}{1!}\,\frac{\frac{\Sigma_{P_3}}{3}}{1!}$

$(\Sigma_{P_1})^2\,\Sigma_{P_5}$.....................*signo* : $(-1)^{2+1+1}$.................*coeficiente* : $\frac{(\Sigma_{P_1})^2}{2!}\,\frac{\frac{\Sigma_{P_5}}{5}}{1!}$

$\Sigma_{P_1}\Sigma_{P_2}\Sigma_{P_4}$.....................*signo* : $(-1)^{1+1+1+1}$...............*coeficiente* : $\frac{\Sigma_{P_1}}{1!}\,\frac{\frac{\Sigma_{P_2}}{2}}{1!}\,\frac{\frac{\Sigma_{P_4}}{4}}{1!}$

$\Sigma_{P_1}(\Sigma_{P_2})^3$.....................*signo* : $(-1)^{1+3+1}$..................*coeficiente* : $\frac{\Sigma_{P_1}}{1!}\,\frac{\left(\frac{\Sigma_{P_2}}{2}\right)^3}{3!}$

$\Sigma_{P_1}(\Sigma_{P_3})^2$.....................*signo* : $(-1)^{1+2+1}$...................*coeficiente* : $\frac{\Sigma_{P_1}}{1!}\,\frac{\left(\frac{\Sigma_{P_3}}{3}\right)^2}{2!}$

$\Sigma_{P_1}\Sigma_{P_6}$.........................*signo* : $(-1)^{1+1+1}$...................*coeficiente* : $\frac{\frac{\Sigma_{P_1}}{1}}{1!}\,\frac{\frac{\Sigma_{P_6}}{6}}{1!}$

$\Sigma_{P_2}\Sigma_{P_5}$.........................*signo* : $(-1)^{1+1+1}$....................*coeficiente* : $\frac{\frac{\Sigma_{P_2}}{2}}{1!}\,\frac{\frac{\Sigma_{P_5}}{5}}{1!}$

$(\Sigma_{P_2})^2\,\Sigma_{P_3}$..................*signo* : $(-1)^{2+1+1}$....................*coeficiente* : $\frac{\left(\frac{\Sigma_{P_2}}{2}\right)^2}{2!}\,\frac{\frac{\Sigma_{P_3}}{3}}{1!}$

$\Sigma_{P_3}\Sigma_{P_4}$........................*signo* : $(-1)^{1+1+1}$....................*coeficiente* : $\frac{\frac{\Sigma_{P_3}}{3}}{1!}\,\frac{\frac{\Sigma_{P_4}}{4}}{1!}$

Σ_{P_7}............................*signo* $:\ (-1)^{1+1}$........................*coeficiente* $:\ \frac{\frac{\Sigma_{P_7}}{7}}{1!}$

IV. SOBRE LA FUNCION $\lambda(z)$.

La función $\lambda(z).= \sum_{n=0}^{n=+\infty} \frac{1}{(2n+1)^z} = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + \ldots\ldots\ldots\ldots\ldots$

cumple la siguiente relación, como demostraré a continuación.

Para todo $z = x + iy, \qquad x,y \in IR, \qquad x > 0$

$\lambda(z).= 1 + \beta(z)\ \lambda(z).$, donde $\beta(z) = \Sigma_{P_1} - \Sigma_2 + \Sigma_3 - \Sigma_4 + \Sigma_5 - \Sigma_6 + \ldots\ldots\ldots\ldots \pm\Sigma_\infty$

Demostración :

Un sumando S_q cualquiera , excepto el 1, de los infinitos que componen a $\lambda(z)$ es de la forma general :

$S_q = \frac{1}{(2n+1)^Z} = \frac{1}{P_1^{Z\,H_1} P_2^{Z\,H_2} P_3^{Z\,H_3}\ldots\ldots P_q^{Z\,Hq}}$ donde $P_1, P_2, P_3, \ldots \in IP - \{2\} \quad n, H_1, H_2, H_3, . \in IN - \{0\}$

Existen exactamente $\binom{Q}{1} = \frac{Q!}{1!(Q-1)!} = Q$ sumandos en Σ_{P_1}, que multiplicados por los

adecuados $\binom{Q}{1}$ sumandos de $\lambda(z)$, generan $\binom{Q}{1}$ veces al sumando S_q. Veamos:

$\frac{1}{P_1^Z} \times \frac{1}{P_1^{Z\,(H_1-1)} P_2^{Z\,H_2} P_3^{Z\,H_3}\ldots\ldots P_q^{Z\,Hq}} = \frac{1}{P_1^{Z\,H_1} P_2^{Z\,H_2} P_3^{Z\,H_3}\ldots\ldots P_q^{Z\,Hq}} = S_q$

$\frac{1}{P_2^Z} \times \frac{1}{P_1^{Z\,H_1} P_2^{Z\,(H_2-1)} P_3^{Z\,H_3}\ldots\ldots P_q^{Z\,Hq}} = \frac{1}{P_1^{Z\,H_1} P_2^{Z\,H_2} P_3^{Z\,H_3}\ldots\ldots P_q^{Z\,Hq}} = S_q$

....

....

$\frac{1}{P_q^Z} \times \frac{1}{P_1^{Z\,H_1} P_2^{Z\,H_2} P_3^{Z\,H_3}\ldots\ldots P_q^{Z\,(Hq-1)}} = \frac{1}{P_1^{Z\,H_1} P_2^{Z\,H_2} P_3^{Z\,H_3}\ldots\ldots P_q^{Z\,Hq}} = S_q$

Existen exactamente $\binom{Q}{2} = \frac{Q!}{2!(Q-2)!} = \frac{Q(Q-1)}{2}$ sumandos en Σ_2 ,que multiplicados

por los adecuados$\binom{Q}{2}$ sumandos de $\lambda(z)$, generan$\binom{Q}{2}$ veces al sumando S_q. Veamos:

$\frac{1}{P_1^Z P_2^Z} \times \frac{1}{P_1^{(H_1-1)Z} P_2^{(H_2-1)Z} P_3^{H_3 Z}\ldots\ldots P_q^{HqZ}} = S_q$

$\frac{1}{P_1^Z P_3^Z} \times \frac{1}{P_1^{(H_1-1)Z} P_2^{H_2 Z} P_3^{(H_3-1)Z}\ldots\ldots P_q^{HqZ}} = S_q$

....

....

$$\frac{1}{P_2^z P_3^z} \times \frac{1}{P_1^{H_1 Z} P_2^{(H_2-1)Z} P_3^{(H_3-1)Z} \ldots\ldots . P_q^{H_q Z}} = S_q$$

$$\frac{1}{P_2^Z\, P_4^Z} \times \frac{1}{P_1^{H_1 Z} P_2^{(H_2-1)Z} P_3^{H_3 Z} P_4^{(H_4-1)Z} \ldots\ldots . P_q^{H_q Z}} = S_q$$

. . . .

. . . .

Del mismo modo, extisten $\binom{Q}{3}$ sumandos en Σ_3 ,que multiplicados por los $\binom{Q}{3}$ sumandos adecuados de $\lambda(z)$ generan $\binom{Q}{3}$ veces al sumando S_q, y así sucesivamente hasta llegar al $\binom{Q}{Q}$ sumando de Σ_q ,que multiplicado por el $\binom{Q}{Q}$ sumando de $\lambda(z)$ genera al sumando S_q.

Veamos:

$$\frac{1}{P_1^{H_1 Z} P_2^{H_2 Z} P_3^{H_3 Z} \ldots\ldots . P_q^{H_q Z}} \times 1 = S_q$$

Por tanto, el sumando cualquiera S_q es generado por el producto $\beta(z)\lambda(z)$ una sóla vez de acuerdo a :

$$\binom{Q}{1} - \binom{Q}{2} + \binom{Q}{3} - \binom{Q}{4} + \ldots\ldots\ldots\ldots \pm \binom{Q}{Q} = \binom{Q}{0} - (1-1)^Q = 1$$

Operamos en la expresión: $\lambda(z). = 1 + \beta(z)\, \lambda(z)$ para obtener:

$$\lambda(z). = \frac{1}{1-\beta(z)} = \frac{1}{\alpha(z)}$$

$$\alpha(z) = 1 - \beta(z) = 1 - \Sigma_{P_1} + \Sigma_2 - \Sigma_3 + \Sigma_4 - \Sigma_5 + \ldots\ldots\ldots\ldots\ldots \mp \Sigma_\infty$$

Utilizando las expresiones generales y las relaciones entre ellas de los apartados I y II, podemos escribir:

$$\alpha(z) = 1 - \Sigma_{P_1} + \left(\frac{\left(\Sigma_{P_1}\right)^2}{2!} - \frac{\frac{\Sigma_{P_2}}{2}}{1!} \right) - \left(\frac{\left(\Sigma_{P_1}\right)^3}{3!} - \frac{\Sigma_{P_1}}{1!}\ \frac{\frac{\Sigma_{P_2}}{2}}{1!} + \frac{\frac{\Sigma_{P_3}}{3}}{1!} \right) +$$

$$+ \left(\frac{\left(\Sigma_{P_1}\right)^4}{4!} - \frac{\left(\Sigma_{P_1}\right)^2}{2!}\ \frac{\frac{\Sigma_{P_2}}{2}}{1!} + \frac{\Sigma_{P_1}}{1!}\ \frac{\frac{\Sigma_{P_3}}{3}}{1!} - \frac{\frac{\Sigma_{P_4}}{4}}{1!} + \frac{\left(\frac{\Sigma_{P_2}}{2}\right)^2}{2!} \right) -$$

$$-\left(\begin{array}{c}\frac{(\Sigma_{P_1})^5}{5!}-\frac{(\Sigma_{P_1})^3}{3!}\frac{\frac{\Sigma_{P_2}}{2}}{1!}+\frac{(\Sigma_{P_1})^2}{2!}\frac{\frac{\Sigma_{P_3}}{3}}{1!}-\frac{\Sigma_{P_1}}{1!}\frac{\frac{\Sigma_{P_4}}{4}}{1!}+\\ +\frac{\frac{\Sigma_{P_5}}{5}}{1!}-\frac{\frac{\Sigma_{P_2}}{2}}{1!}\frac{\frac{\Sigma_{P_3}}{3}}{1!}+\frac{\Sigma_{P_1}}{1!}\frac{\left(\frac{\Sigma_{P_2}}{2}\right)^2}{2!}\end{array}\right)+\ldots\ldots\ldots\ldots\infty$$

que operando y reagrupando términos puede expresarse así:

$$\alpha(z)=\left(1-\frac{\Sigma_{P_1}}{1!}+\frac{(\Sigma_{P_1})^2}{2!}-\frac{(\Sigma_{P_1})^3}{3!}+\ldots\ldots\ldots\right)\times\left(1-\frac{\frac{\Sigma_{P_2}}{2}}{1!}+\frac{\left(\frac{\Sigma_{P_2}}{2}\right)^2}{2!}-\frac{\left(\frac{\Sigma_{P_2}}{2}\right)^3}{3!}+\ldots\ldots\ldots\right)$$

$$\times\left(1-\frac{\frac{\Sigma_{P_3}}{3}}{1!}+\frac{\left(\frac{\Sigma_{P_3}}{3}\right)^2}{2!}-\frac{\left(\frac{\Sigma_{P_3}}{3}\right)^3}{3!}+\ldots\ldots\ldots\right)\times\left(1-\frac{\frac{\Sigma_{P_4}}{4}}{1!}+\frac{\left(\frac{\Sigma_{P_4}}{4}\right)^2}{2!}-\frac{\left(\frac{\Sigma_{P_4}}{4}\right)^3}{3!}+\ldots\ldots\ldots\right)$$

$$\times\times\times\ldots\ldots\ldots\ldots\ldots\ldots\infty=\left(\sum_{n=0}^{n=\infty}\frac{(-1)^n\left(\frac{\Sigma_{P_1}}{1}\right)^n}{n!}\right)\left(\sum_{n=0}^{n=\infty}\frac{(-1)^n\left(\frac{\Sigma_{P_2}}{2}\right)^n}{n!}\right)\left(\sum_{n=0}^{n=\infty}\frac{(-1)^{n-1}\left(\frac{\Sigma_{P_3}}{3}\right)^n}{n!}\right)\ldots$$

$$=\prod_{p=3}^{p=\infty}\left(\sum_{n=0}^{n=\infty}\frac{(-1)^n\left(\frac{\Sigma_{P_m}}{m}\right)^n}{n!}\right)=e^{-\Sigma_{P_1}}\;e^{-\frac{\Sigma_{P_2}}{2}}\;e^{-\frac{\Sigma_{P_3}}{3}}\ldots\ldots\infty=e^{-\left(\Sigma_{P_1}+\frac{\Sigma_{P_2}}{2}+\frac{\Sigma_{P_3}}{3}+\ldots\infty\right)}=$$

$$=e^{-\left(\sum_{P=3}^{P=\infty}\ln\frac{P^Z}{P^Z-1}\right)}=e^{-\ln\left(\prod_{p=3}^{p=\infty}\frac{P^Z}{P^Z-1}\right)}$$

obteniéndose finalmente $\lambda(z).=\frac{1}{1-\beta(z)}=\frac{1}{\alpha(z)}=e^{\ln\left(\prod_{p=3}^{p=\infty}\frac{P^Z}{P^Z-1}\right)}=\prod_{p=3}^{p=\infty}\frac{P^Z}{P^Z-1}$

y multiplicando en ambos miembros de la igualdad anterior por $\frac{2^Z}{2^Z-1}$ obtenemos el producto de Euler:

$$\frac{2^Z}{2^Z-1}\lambda(z)=\sum_{n=1}^{n=\infty}\frac{1}{n^Z}=\frac{2^Z}{2^Z-1}\left(\prod_{p=3}^{p=\infty}\frac{P^Z}{P^Z-1}\right)=\prod_{p=2}^{p=\infty}\frac{P^Z}{P^Z-1}$$

Ha realizado este trabajo Niceto Valcárcel Yeste, Licenciado en CC Físicas por la U.N.E.D.

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